

The Measurement-Disturbance Relation and the Disturbance Trade-off Relation in Terms of Relative Entropy

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We employ quantum relative entropy to establish the relation between the measurement uncertainty and its disturbance on a state in the presence (and absence) of quantum memory. For two incompatible observables, we present the measurement-disturbance relation and the disturbance trade-off relation. We find that without quantum memory the disturbance induced by the measurement is never less than the measurement uncertainty and with quantum memory they depend on the conditional entropy of the measured state. We also generalize these relations to the case with multiple measurements. These relations are demonstrated by two examples.

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I. Introduction

The Heisenberg uncertainty principle is one of the well-known fundamental principles in quantum mechanics [1]. It comes from a thought experiment about the position measurement of a particle by using a γ -ray microscope. The result shows that anyone is not able to specify the values of the non-commuting and canonically conjugated variables simultaneously. Later Kennard-Robertson inequality extended Heisenberg uncertainty principle to arbitrary pairs of observables based on the variance [2, 3]

$$\Delta X \Delta Y \geq \frac{1}{2} |\langle \Psi | [X, Y] | \Psi \rangle|, \quad (1)$$

where $\Delta X = \sqrt{\langle \Psi | (X - \langle X \rangle)^2 | \Psi \rangle}$ represents the variance of the observable X and $[X, Y] = XY - YX$ stands for the commutator. The inequality (1) describes the limitations on our ability to simultaneously predict the measurement outcomes of non-commuting observables in quantum theory. One can see that the lower bound of the Robertson's relation is determined by the wave-function and the commutator of the observables. So the Robertson's relation could arrive at a trivial bound if $|\psi\rangle$ leads to the zero expectation value of the commutator.

With the development of the modern quantum mechanics, a variety of uncertainty relations were proposed from different angles [4–7]. From the point of informatics of view, in 1983, Deutsch [8] presented the uncertainty relation for the conjugate observables based on the Shannon entropy. Subsequently, Kraus [9] gave a stronger conjecture of the uncertainty relation and Maassen and Uffink proved it in a succinct form as [10]

$$H(X) + H(Y) \geq -\log c, \quad (2)$$

where $c = \max_{i,j} |\langle x_i | y_j \rangle|^2$ quantifies the complementarity of the non-degenerate observables X and Y with $|x_i\rangle, |y_j\rangle$ denoting their eigenvectors and $H(X)$ ($H(Y)$) is the Shannon entropy

of the probability distribution corresponding to the outcomes of the observable X (Y). It is obvious that this lower bound given in Eq. (2) doesn't depend on the state to be measured. Meanwhile, the uncertainty relations have been generalized for more than two observables [11–13]. Based on different definitions of entropy, various entropic uncertainty relations have been presented [14–38] and many relevant works have been summarized in the review article [6]. From the point of geometry of view, the Landau-Pollak uncertainty relation has been proposed in terms of maximum probabilities for the measurement outcomes of the two observables [10, 39, 40]. Considering the quantum uncertainty relation widely used in the quantum information processing, in particular, the direct application in quantum key distribution, Berta *et al.* [41] generalized entropic uncertainty relation to the case in the presence of quantum memory, that is,

$$H(X|B) + H(Y|B) \geq -\log c + H(A|B), \quad (3)$$

where the quantum conditional entropy $H(X|B) = H(\rho_{XB}) - H(\rho_B)$ with ρ_{XB} denoting the state after X measurement on subsystem A of ρ_{AB} and $H(\rho)$ is the von Neumann entropy of the quantum state ρ . In addition, as is known to all, quantum state is usually destroyed by measurements due to the measurement-induced collapse of the state. So there usually exist disturbances between the quantum states before and after the measurements. It is shown that, similar to the uncertainty relation which mainly bounds the incompatible measurements (We refer to the measurement outcomes and the corresponding probability distribution), there also exist the limitations on the disturbance induced by incompatible measurements (D-D relation), and the limitations on the disturbance induced by one measurement and the uncertainty of the other measurement (M-D relation) [42–46].

In this paper, we study the measurement-measurement uncertainty (M-M) relation, the measurement-disturbance (M-D) relation and the disturbance tradeoff (D-D) relation in the presence (and absence) of quantum memory. We mainly establish the lower bounds on M-D relation and D-D relation. It is especially interesting that if there exists quantum memory we find that the measurement uncertainty is closely related to its disturbance by the conditional entropy of the measured state; if there is no quantum memory, it is related

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to each other just by the uncertainty of the measured state itself. In this sense, we obtain a conclusion that the disturbance is never less than the measurement uncertainty if there is no quantum memory. The paper is organized as follows. In Sec. II, we mainly consider the M-M relation, M-D relation and D-D relation in the presence of quantum memory and study the relationship among three relations. In Sec. III, we present these relations for a pair of the incomparable measurement in the absence of quantum memory. In Sec. IV, we generalize our results to the cases of multiple measurements. Finally, we draw our conclusion.

II. Uncertainty relations in the presence of quantum memory

To begin with, we introduce the rules of the quantum games similar to the scenario of Ref. [41]. Suppose Alice and Bob share a bipartite state ρ_{AB} with qubits A and B at Alice's and Bob's hand respectively. Beforehand, Alice and Bob agree on two measurements Π^1 and Π^2 with $\{\Pi_k^1 = |\pi_k^1\rangle\langle\pi_k^1|\}$ and $\{\Pi_l^2 = |\pi_l^2\rangle\langle\pi_l^2|\}$ denoting the projectors of the corresponding eigenvectors. Alice performs either measurement Π^1 or Π^2 on her qubit A and announce her measurement outcomes. Bob tries his best to minimize his uncertainty about Alice's measurement outcomes with the assistance of his qubit B.

Let's first only consider a single observable Π^j . Let $\{p_k^j\}$ with $p_k^j = \text{tr}(\Pi_k^j \rho_{AB})$ denote the probability distribution of the j th measurement. The post-measurement quantum state can be given by $\rho_{\Pi^j B} = \sum_k (\Pi_k^j \otimes \mathbb{I}) \rho_{AB} (\Pi_k^j \otimes \mathbb{I}) / p_k^j$. Therefore, the measurement uncertainty of Π^j in the presence of the quantum memory is described by the conditional entropy $H(\Pi^j|B) = H(\rho_{\Pi^j B}) - H(\rho_B)$. Since measurements could lead to the collapse of quantum state, the final state $\rho_{\Pi^j B}$ is usually different from the initial state ρ_{AB} . The disturbance induced by such a measurement can be characterized by the 'distance' between them. Here we would like to employ quantum relative entropy as the 'distance' measure, so the disturbance can be defined by $H(\rho_{AB} \|\rho_{\Pi^j B})$. It is obvious that if a non-demolition measurement is performed or the state is not disturbed, the measured state is not disturbed. So the disturbance is zero. Here one can find that the uncertainty is closely related to the disturbance, which can be given in the following rigorous form.

Theorem.1. The trade-off relation between the measurement uncertainty and its disturbance on the state ρ_{AB} can be given by

$$H(\rho_{AB} \|\rho_{\Pi^j B}) - H(\Pi^j|B) = -H(A|B). \quad (4)$$

Proof. Based on the definition of quantum relative entropy

$$\begin{aligned} & H(\rho_{AB} \|\rho_{\Pi^j B}) \\ &= \text{tr} \rho_{AB} \log \rho_{AB} - \text{tr} \rho_{AB} \log \rho_{\Pi^j B} \\ &= -H(\rho_{AB}) - \text{tr} \rho_{AB} \log \sum_m (|\pi_m^j\rangle\langle\pi_m^j| \otimes \mathbb{I}) \rho_{AB} (|\pi_m^j\rangle\langle\pi_m^j| \otimes \mathbb{I}) \\ &= -H(\rho_{AB}) + H(\rho_B) - H(\rho_B) + H(\rho_{\Pi^j B}) \\ &= -H(A|B) + H(\Pi^j|B). \\ &\Rightarrow H(\rho_{AB} \|\rho_{\Pi^j B}) - H(\Pi^j|B) = -H(A|B). \end{aligned} \quad (5)$$

The proof is finished. \blacksquare

Eq. (4) shows that the disturbance and the measurement uncertainty are connected by the conditional entropy of the measured state. Intuitively, the measured state will be greatly disturbed, and the measurement uncertainty is greatly reduced if the subsystem A and B are strongly entangled [41]. Their gap is compensated for by the conditional entropy of the original state which embodies the entanglement between A and B to some extent. This can be easily understood if the original state is a maximally entangled state (e.g. a Bell state with -1 original conditional entropy and vanishing post-measurement conditional entropy) [47, 48]. In addition, one can find that, if $H(A|B) > 0$, the measurement uncertainty is larger than its disturbance on the state; if $H(A|B) < 0$, the measurement uncertainty is less than its disturbance on the state; and they are equal for $H(A|B) = 0$. In other words, they strongly depend on $H(A|B)$.

Now let's turn back to the game with two incompatible observables Π^1 or Π^2 , the total measurement uncertainties given by $H(\Pi^1|B) + H(\Pi^2|B)$ which, as we know, are bounded by the inequality (3) with X and Y replaced by Π^1 and Π^2 and $c = \max_{k,l} |\langle\pi_k^1|\pi_l^2\rangle|^2$. This is the M-M relation. Here we would like to find the bound on the measurement uncertainty of one observable and the disturbance induced by the other observable, i.e., the M-D relation; and the bound on the disturbances induced by two incompatible observables, i.e., the D-D relation. Based on our theorem.1 and Eq. (3), one can easily find the following theorem.

Theorem.2. For the projective measurements of the two observables Π^1 and Π^2 on quantum state ρ_{AB} , in the presence of the quantum memory, the M-D relation is given by

$$H(\rho_{AB} \|\rho_{\Pi^1 B}) + H(\Pi^2|B) \geq -\log c, \quad (6)$$

or

$$H(\rho_{AB} \|\rho_{\Pi^2 B}) + H(\Pi^1|B) \geq -\log c, \quad (7)$$

and the D-D relation is given by

$$H(\rho_{AB} \|\rho_{\Pi^1 B}) + H(\rho_{AB} \|\rho_{\Pi^2 B}) \geq -\log c - H(A|B), \quad (8)$$

with $c = \max_{k,l} |\langle\pi_k^1|\pi_l^2\rangle|^2$.

Proof. Omitted. \blacksquare

From the above theorem, the relationship among the M-M, M-D and D-D relations is illustrated in Fig. 1 (a). It shows that the three relations can be converted into each other by

considering the contribution of $H(A|B)$. But we should note that they reveal different physics.

To demonstrate the relationship among these three relations, we take the Werner state as the measured state to be an example. The Werner state is given by [49]

$$\rho_{AB} = \eta |\psi^\dagger\rangle\langle\psi^\dagger| + \frac{1-\eta}{4}\mathbb{I}, \quad (9)$$

with $|\psi^\dagger\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ the maximally entangled state and the purity denoted by η , $0 \leq \eta \leq 1$. The two incompatible observables Π^1 and Π^2 are performed on subsystem A. The eigenvectors of Π^1 is given by

$$\Pi^1 : \left\{ \left(\cos \frac{\theta}{2}, -e^{i\phi} \sin \frac{\theta}{2} \right), \left(e^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right) \right\}. \quad (10)$$

with the azimuthal angle $0 \leq \phi \leq 2\pi$ and the polar angle $0 \leq \theta \leq \pi$. Similarly, the other projective measurement related to Π^2 is defined by

$$\Pi^2 : \left\{ \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \right\}. \quad (11)$$

A straightforward computation gives the quantum conditional entropy of the initial state ρ_{AB}

$$H(A|B) = -\frac{3(1-\eta)}{4} \log \frac{(1-\eta)}{4} - \frac{(1+3\eta)}{4} \log \frac{(1+3\eta)}{4} - 1. \quad (12)$$

So the total measurement uncertainty (of Π^1 and Π^2) reads

$$H(\Pi^1|B) + H(\Pi^2|B) = 2 - \sum_{m=\pm 1} (1+m\eta) \log(1+m\eta), \quad (13)$$

the one uncertainty plus the other disturbance is

$$\begin{aligned} & H(\rho_{AB} \| \rho_{\Pi^1 B}) + H(\Pi^2|B) \\ &= 1 + \frac{1+3\eta}{4} \log(1+3\eta) - \frac{1-\eta}{4} \log(1-\eta) \\ & - (1+\eta) \log(1+\eta) = H(\rho_{AB} \| \rho_{\Pi^2 B}) + H(\Pi^1|B), \end{aligned} \quad (14)$$

and the total disturbance is

$$\begin{aligned} & H(\rho_{AB} \| \rho_{\Pi^1 B}) + H(\rho_{AB} \| \rho_{\Pi^2 B}) \\ &= \frac{1+3\eta}{2} \log(1+3\eta) + \frac{1-\eta}{2} \log(1-\eta) - (1+\eta) \log(1+\eta). \end{aligned} \quad (15)$$

In Fig.1 (b), we plot the results of Eqs. (13-15). It presents that $H(\Pi^1|B) + H(\Pi^2|B) \geq H(\rho_{AB} \| \rho_{\Pi^1 B}) + H(\Pi^2|B) \geq H(\rho_{AB} \| \rho_{\Pi^2 B}) + H(\Pi^1|B)$ below $\eta \approx 0.7476$; they are equal at the critical point $\eta \approx 0.7476$; and the relations will be reverse if η is beyond this critical point.

III. Uncertainty relations in the absence of quantum memory

When there is not any quantum memory, that means that the subsystems A and B have nothing with each other. So we

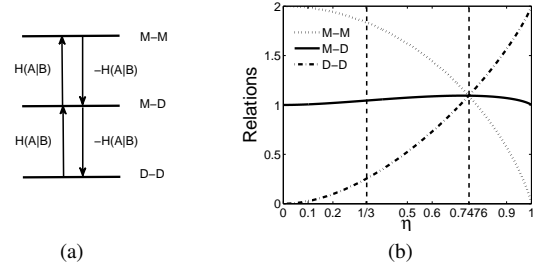


FIG. 1. (a) The relationship among the M-M, M-D, D-D in the presence of quantum memory. (b) M-M, M-D, D-D vs. η . For the projective measurements the azimuthal angle $\phi = 0$ and the polar angle $\theta = \pi/2$, that is, $\Pi^1 : \left\{ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$. The dot line stands for the M-M, the line stands for the M-D and the dash-dot line stands for the D-D.

can set $\tilde{\rho}_{AB} = \rho \otimes \rho_B$. Substitute $\tilde{\rho}_{AB}$ into Eq. (5), one will immediately obtain that

$$H(\rho \| \rho_{\Pi^i}) = H(\rho_{\Pi^i}) - H(\rho). \quad (16)$$

Eq. (16) shows that the measurement uncertainty includes two parts. One is the disturbance induced by the measurement, the other is the uncertainty of the measured state itself. From a different angle, one can also find that the disturbance of a measurement is just the entropy increment of the post- and pre-measurement states. Since $H(\rho) \geq 0$ for any ρ , an important conclusion is that the disturbance by a measurement is never less than its measurement uncertainty. They are equal if and only if ρ is a pure state.

If we substitute Eq. (16) into Eqs. (6), (7) and (8), one can find that the M-M relation, M-D relation and D-D relation in the absence of quantum memory as follows.

Corollary.1. For the projective measurements of the two observables Π^1 and Π^2 on quantum state ρ in the absence of quantum memory, the M-M relation is given by [10]

$$H(\rho_{\Pi^1}) + H(\rho_{\Pi^2}) \geq -2 \log c + H(\rho), \quad (17)$$

the M-D relation reads

$$H(\rho \| \rho_{\Pi^1}) + H(\rho_{\Pi^2}) \geq -\log c, \quad (18)$$

or

$$H(\rho \| \rho_{\Pi^2}) + H(\rho_{\Pi^1}) \geq -\log c, \quad (19)$$

and the D-D relation is

$$H(\rho \| \rho_{\Pi^1}) + H(\rho \| \rho_{\Pi^2}) \geq -\log c - H(\rho), \quad (20)$$

where $c = \max_{k,l} \left| \langle \pi_k^1 | \pi_l^2 \rangle \right|^2$.

Proof. Omitted. ■

Considering Eq. (16), one can also find that the three relations given in Corollary 1 can be converted into each other. Especially, if we consider Eq. (16) for both observables, one will trivially obtain

$$\begin{aligned} & H(\rho_{\Pi^1}) + H(\rho_{\Pi^2}) \\ &= H(\rho \| \rho_{\Pi^1}) + H(\rho_{\Pi^2}) + H(\rho) \\ &= H(\rho \| \rho_{\Pi^1}) + H(\rho \| \rho_{\Pi^2}) + 2H(\rho). \end{aligned} \quad (21)$$

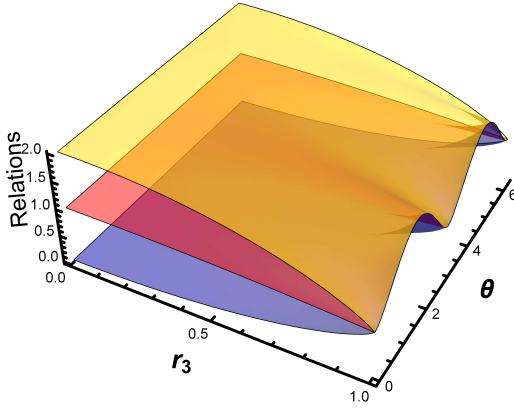


FIG. 2. (color online) The M-M, the M-D the D-D for the two 2-dimensional projective measurements vs. r_3 and θ in presence of quantum memory. The yellow curved surface stands for the M-M, the red curved surface stands for the M-D and the blue curved surface stands for the D-D.

This also states that the total disturbances are never less than the measurement uncertainties. From the viewpoint of Heisenberg uncertainty principle of view, the incompatible observables cannot be simultaneously accurately determined since the measurements always lead to the disturbance.

As an example, we compare the total uncertainties, the one uncertainty plus the other disturbance and the total disturbances in the absence of quantum memory. First, we choose the arbitrary single qubit $\rho = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with the standard Pauli operators $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ and $\sum_i r_i^2 = 1$ as the measured state. The incompatible observables X and Y can be taken the same as Eqs. (10,11). Through a simple calculation, we can obtain the following expressions:

$$\begin{aligned} & H(\rho_X) + H(\rho_Y) \\ &= \sum_{m=\pm 1} -\frac{1+mr_3 \cos \theta}{2} \log \frac{1+mr_3 \cos \theta}{2} \\ & \quad -\frac{2+mr_3}{4} \log \frac{2+mr_3}{4}, \end{aligned} \quad (22)$$

$$\begin{aligned} & H(\rho||\rho_X) + H(\rho||\rho_Y) \\ &= \sum_{m=\pm 1} \frac{1+mr_3}{2} \log \frac{1+mr_3}{2} - \frac{2+mr_3}{4} \log \frac{2+mr_3}{4} \\ & \quad -\frac{1+mr_3 \cos \theta}{2} \log \frac{1+mr_3 \cos \theta}{2}, \end{aligned} \quad (23)$$

and

$$\begin{aligned} & H(\rho||\rho_X) + H(\rho||\rho_Y) \\ &= \sum_{m=\pm 1} (1+mr_3) \log \frac{1+mr_3}{2} - \frac{2+mr_3}{4} \log \frac{2+mr_3}{4} \\ & \quad -\frac{1+mr_3 \cos \theta}{2} \log \frac{1+mr_3 \cos \theta}{2}. \end{aligned} \quad (24)$$

In Fig.2, we plot the results of Eqs. (22-24). It shows three layers which stand for our mentioned relations. The upper curved surface (yellow) stands for the M-M whilst the lower

one (blue) is the D-D. In between, the curved surface is M-D. If $r_3 = 1$, the initial quantum state reduce to a pure state $\rho = |0\rangle\langle 0|$, i.e., the von Neuman entropy is zero, $H(\rho) = 0$, they are well consistent.

IV. Uncertainty relations for the multiple measurements in the presence (and absence) of the quantum memory

In this section, we extend the M-D relation and the D-D relation to the multiple measurements in the presence (and absence) of quantum memory. In order to give a clear background, we would like to describe the game at first. Suppose that Alice performs a group of measurements $\{\Pi_i, i = 1, 2, \dots, N\}$ on a state ρ_{AB} . Alice chooses one measurement Π_i and announces her choice to Bob. Bob tries to minimize his uncertainty about Alice's measurement outcomes. During this game, for the multiple measurements acted on the initial quantum state, $\{\Pi^i, i = 1, 2, \dots, N\}$ can be rearranged in different orders with ε labelling the different orders. Thus, Π^{ε_i} can be understood as i th measurement in the ε order. Similarly, the α th eigenvector of Π^{ε_i} can be written as $|\varepsilon_i^\alpha\rangle$. With all the above knowledge, the state-dependent entropic uncertainty relation will be given by [13].

$$\sum_{i=1}^N H(\Pi_i|B) \geq \max \{ \mathcal{L}_1, \mathcal{L}_{opt}, 0 \}. \quad (25)$$

with

$$\mathcal{L}_1 = (N-1)H(A|B) + \max_{\varepsilon} \{ \ell_{\varepsilon}^U \}, \quad (26)$$

$$\mathcal{L}_{opt} = \frac{N}{2}H(A|B) + \max_{all \text{ ways}} \mathcal{B}'_{ways}, \quad (27)$$

where

$$\ell_{\varepsilon}^U = - \sum_{\alpha_N} p_{\varepsilon_N^{\alpha_N}} \log \sum_{\alpha_k, N \geq k > 1} \max_{\alpha_1} \prod_{n=1}^{N-1} \left| \langle \varepsilon_n^{\alpha_n} | \varepsilon_{n+1}^{\alpha_{n+1}} \rangle \right|^2, \quad (28)$$

with $p_{\varepsilon_N^{\alpha_N}} = \text{Tr}(|\varepsilon_N^{\alpha_N}\rangle \langle \varepsilon_N^{\alpha_N}| \otimes \mathbb{I}) \rho_{AB}$ and \mathcal{B}'_{ways} is average value of $-\sum_{\alpha_2} p_{\varepsilon_2^{\alpha_2}} \log \max_{\alpha_1} \left| \langle \varepsilon_1^{\alpha_1} | \varepsilon_2^{\alpha_2} \rangle \right|^2$ for all potential two-measurement combinations, that is ℓ_{ε}^U is constrained for only two measurements.

Suppose that the measurement uncertainties come from γ measurements while the disturbances come from β measurements with $\gamma + \beta = N$. Then, the M-D relation and the D-D relation for the multiple measurements in the presence (and absence) of quantum memory will be given as follows.

Corollary.2. In the presence of quantum memory, let the initial quantum state ρ_{AB} be measured by the set of observables $\{\Pi^i, i = 1, 2, \dots, N\}$. The state-dependent trade-off relation between the γ measurements uncertainties and the β disturbances is given by

$$\gamma \sum_{i=1}^{\gamma} H(\Pi^i|B) + \beta \sum_{j=1}^{\beta} H(\rho_{AB}||\rho_{\Pi^j B}) \geq \max \{ \mathcal{L}'_1, \mathcal{L}'_{opt}, 0 \}, \quad (29)$$

with

$$\mathcal{L}'_1 = (N - 1 - \beta)H(A|B) + \max_{\varepsilon} \{\ell_{\varepsilon}^U\}, \quad (30)$$

$$\mathcal{L}'_{opt} = (\frac{N}{2}H - \beta)(A|B) + \max_{all\ ways} \mathcal{B}'_{ways}. \quad (31)$$

The state-dependent trade-off relation in the absence of quantum memory is given by

$$\gamma \sum_{i=1}^{\gamma} H(\rho_{\Pi^i}) + \beta \sum_{j=1}^{\beta} H(\rho_{\Pi^j}) \geq \max \{\mathcal{L}''_1, \mathcal{L}''_{opt}, 0\}, \quad (32)$$

with

$$\mathcal{L}''_1 = (N - 1 - \beta)H(\rho) + \max_{\varepsilon} \{\ell_{\varepsilon}^U\}, \quad (33)$$

$$\mathcal{L}''_{opt} = (\frac{N}{2}H - \beta)H(\rho) + \max_{all\ ways} \mathcal{B}'_{ways}, \quad (34)$$

where ℓ_{ε}^U is given by Eq. (28).

Proof. Omitted ■

It is obvious that the term ℓ_{ε}^U which describes the complementarity of the non-degenerate observables depends on the sequence of observables and the initial quantum state. In order to eliminate the state dependency, we take maximum over α_N of Π^{ε_N} , so ℓ_{ε}^U in the second term becomes

$$\begin{aligned} \ell_{\varepsilon}^U &= - \sum_{\alpha_N} p_{\varepsilon_N^{\alpha_N}} \log \sum_{\alpha_k, N \geq k > 1} \max_{\alpha_1} \prod_{n=1}^{N-1} |\langle \varepsilon_n^{\alpha_n} | \varepsilon_{n+1}^{\alpha_{n+1}} \rangle|^2 \\ &\geq - \max_{\alpha_N} \log \sum_{\alpha_k, N \geq k > 1} \max_{\alpha_1} \prod_{n=1}^{N-1} |\langle \varepsilon_n^{\alpha_n} | \varepsilon_{n+1}^{\alpha_{n+1}} \rangle|^2 = \ell_{\varepsilon}^{\tilde{U}}. \end{aligned} \quad (35)$$

Replacing ℓ_{ε}^U with $\ell_{\varepsilon}^{\tilde{U}}$ in Eqs. (29,32), we can obtain the corresponding state-independent trade-off relation between the γ measurements uncertainties and the β disturbances in the presence (and absence) of quantum memory. From the Corollary.2, we can choose $\gamma = N, \beta = 0$ measurements to establish

the M-M relation whilst choose $\gamma = 0, \beta = N$ measurements to establish the D-D relation. At the same time the relationship among the M-M, the M-D and the D-D for the multiple measurements is similar to the relationship for two incompatible measurements.

V. Conclusion and Discussion

We present the relation between the measurement uncertainty and its disturbance in terms of quantum relative entropy. We find that in the presence of quantum memory, the measurement uncertainty is closely related to its disturbance on the measured state by the conditional entropy of the original state, and in the absence of quantum memory, the disturbance of a measurement can be described by its measurement uncertainty plus the uncertainty of the measured state. In other words, the disturbance can be understood by the entropic increment of the post- and pre- measurement state. Based on such relations, we study the measurement-measurement uncertainty relation, the measurement-disturbance relation and the disturbance tradeoff relation in the presence (and absence) of quantum memory. At the same time, we also establish the relationship among the M-M, D-D and M-D in the presence (absence) of quantum memory. Our results have also been extended to the case with multiple measurements. Finally, we also demonstrate these relations by concrete example.

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Relations

